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# A CANONICAL STRUCTURES OF THE STANDARD CLIFFORDIAN *Kähler* MANIFOLDS ON THE TANGENT SPACE AND THE COTANGENT SPACE

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**ABSTRACT**

In this paper we study the a canonical structure on the tangent space and the cotangent space of the Standard Cliffordian Kähler Manifolds, a canonical structures of the tangent bundle $TM$ and a canonical structures of the cotangent bundle $T^*M$ is called the standard Cliffordian $Kähler$ Manifolds.



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# INTRODUCTION

Differential geometry and complex geometry are vital fields in modern mathematics, enabling the study of multi-dimensional spaces through their associated algebraic and geometric structures. Among these spaces manifolds that integrate Kähler geometry with Clifford algebra attract significant interest due to their rich symmetrical and physical properties. In this context, this paper focuses on the study of standard cliffordian Kähler manifolds, specifically by examining the canonical structures defined on both the tangent space and the cotangent space. Accordingly, a space is referred to a " Standard Cliffordian Kähler Manifolds" when a specific canonical structure exists on the tangent bundle $TM$, in tandem with a canonical structure on the cotangent bundle $T^*M$.

**Almost Complex Structures**

Let $M$ be a differentiable manifold of class $C^\infty$ and dimension $m \geq 2$.

If there exists on $M$ a tensor $J_i^j$ of Type (1,1) satisfying:

$$J_i^j J_j^k = -\mathcal{E}_i^k \qquad \rightarrow \qquad (1)$$

Where the indices $i\,, j\,, k = 1, \ldots, m$ the repeated indices imply summation, and $\mathcal{E}_i^k$ are Kronecker symbols defined by:

$$\mathcal{E}_i^k = \begin{cases} 1 & , \quad i = k \\ 0 & , \quad i \neq k \end{cases} \qquad \rightarrow \qquad (2)$$

Then $J_i^j$ is said to define an almost complex structure on M, and the Manifold M is called an almost complex Manifold.

For simplicity we shall call the almost complex structure defined by $J_i^j$ the almost complex structure $J_i^j$. [1]

**The Tangent and Cotangent Bundles:** We define the tangent bundles of a manifold M as the (disjoint) union of the tangent spaces ; $TM = U_{p\in M} T_p M$.

**Definition**

Given a smooth map $f: M \to N$ as above , the tangent maps $T_p f$ on the individual tangent spaces combine to give a map:

$$Tf : TM \to TN \qquad \to \tag{3}$$

On the tangent bundle which is linear on each fiber. This map is called the tangent map or sometimes the tangent lift of $f$.

For smooth maps $f: M \to N \; and \; g: N \to M$ we have the following simple looking version of the chain rule:

$$T(gof) = TgoTf \qquad \to \tag{4}$$

If $U$ is an open set in a finite-dimensional vector space $V$, then the tangent space at $x \in U$ can be viewed as $\{x\} \times V$. For example, recall that an element $V_p = (p, V)$ corresponds to the derivation:

$$f \to V_p f := \left.\frac{d}{dt}\right|_{t=0} f(p + tv) \qquad \to \tag{5}$$

Thus the tangent bundle of $U$ can be viewed as the product $U \times V$.

Let $U_1 \; and \; U_2$ be open subsets of vector spaces $V \; and \; W$ respectively and let $f : U_1 \to U_2$ be smooth ( or at least $\acute{C}$ ). Then we have the tangent map $Tf : TU_1 \to TU_2$.

Viewing $TU_1 \; as \; U_1 \times V$ and similarly for $U_2$ , the tangent map $Tf$ is given by $(p , v) \to ( f(p), Df(p).v)$.

**Definition** Define the cotangent bundle of a manifold M to be the set:

$$T^*M := \cup_{p\in M} T_p^* M \qquad \to \tag{6}$$

And define the map $\pi_{T^*M} : T^*M \to M$ to be the obvious projection taking elements in each space $T_p^* M$ to the corresponding point $p$. [2]

**Cliffordian $\boldsymbol{K\ddot{a}hler}$ Manifold**

Here, we recalled the main concepts and structures given in [3,4,5]. Let $M$ be a real smooth manifold of dimension $m$. Suppose that there is a 6-dimensional vector bundle $V$ consisting of $F_i (i = 1,2, \dots ,6)$ tensors of type(1,1) over $M$. Such a local basis $\{F_1, F_2, \dots , F_6\}$ is called a canonical local basis of the bundle $V$ in neighborhood $U \; of \; M$. Then $V$ is called an almost Cliffordian structure in $M$. The pair $(M, V)$ is named an almost Cliffordian manifold with $V$. Hence, an almost Cliffordian manifold $M$ is of $dimension \; m = 8n$. If there exists on $(M, V)$ a global basis $\{F_1, F_2, \dots , F_6\}, then \; (M, V)$ is said to be an almost Cliffordian manifold; the basis $\{F_1, F_2, \dots , F_6\}$ is called a global basis for $V$. An almost Cliffordian connection on the almost Cliffordian manifold$(M, V)$ is a linear connection $\nabla \; on \; M$ which preserves by parallel transport the vector bundle $V$. This means that if $\Phi$ is a cross-section (local-global) of the bundle $V , then \; \nabla_X \Phi$ is also a cross-section (local-global, respectively) of $V , X$ being an arbitrary vector field of $M$.

If for any canonical basis $\{J_1, J_2, \dots , J_6\} \; of \; V$ in a coordinate neighborhood , the identities

$$g(J_i X , J_i Y) = g(X , Y), \; \forall X, Y \in \mathcal{X}(M) , \; i = 1,2, \dots ,6 \qquad \to \tag{7}$$

Hold, the triple $(M, g, V)$ is named an almost Cliffordian Hermitian manifold or metric Cliffordian denoting by $V$ an almost Cliffordian structure $V \; and \; by \; g$ a Riemannian metric and by $(g, V)$ an almost Cliffordian metric structure.

Since each $J_i (i = 1,2, \dots ,6)$ is almost Hermitian structure with respect to $g$, setting

$$\Phi_i (X, Y) = g(J_i X, Y), \quad i = 1,2, \dots ,6 \qquad \to \tag{8}$$

For any vector fields $and \; Y$ , we see that $\Phi_i$ are 6 local 2-forms. If the Levi-Civita connection $\nabla = \nabla^g \; on \; (M, g, V)$ preserves the vector bundle $V$ by parallel transport, then $(M, g, V)$ is called a Cliffordian $K\ddot{a}hler$ manifold, and an almost Cliffordian structure $\Phi_i \; of \; M$ is called a Cliffordian $K\ddot{a}hler$ structure. A Clifford $K\ddot{a}hler$ manifold is Riemannian manifold $(M^{8n}, g)$.

For example, we say that $R^{8n}$ is the simplest example of Clifford $K\ddot{a}hler$ manifold. Suppose that let

$$\{x_i \,, x_{i+n} \,, x_{i+2n} \,, x_{i+3n} \,, x_{i+4n} \,, x_{i+5n} \,, x_{i+6n} \,, x_{i+7n}\} \,, i = \overline{1,n}$$

be a real coordinate system on $R^{8n}$.

The frame field represents the natural bases over $R$ of the tangent space $T(R^{8n}) of\ R^{8n}$ and can be written:

$$\left\{\frac{\partial}{\partial x_{an+i}}\right\} \quad , \quad a = 0,1,2,3, \ldots 7 \quad \rightarrow \tag{9}$$

The co-frame field represents the natural bases over $R$ of the cotangent space $T^*(R^{8n})\ of\ R^{8n}$ and can be written:

$$\{dx_{an+i}\} \quad , \quad a = 0,1,2,3, \ldots ,7 \quad \rightarrow \tag{10}$$

By a canonical local basis $\{J_1 \,, J_2 \,, J_3\}\ of\ V$ of the tangent space $TM$ of Manifold M, the following expressions are obtained

$$\begin{array}{lll}
J_1\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{n+i}} & J_2\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{2n+i}} & J_3\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{3n+i}} \\
J_1\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_i} & J_2\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_{4n+i}} & J_3\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_{5n+i}} \\
J_1\left(\frac{\partial}{\partial x_{2n+i}}\right) = \frac{\partial}{\partial x_{4n+i}} & J_2\left(\frac{\partial}{\partial x_{2n+i}}\right) = -\frac{\partial}{\partial x_i} & J_3\left(\frac{\partial}{\partial x_{2n+i}}\right) = -\frac{\partial}{\partial x_{6n+i}} \\
J_1\left(\frac{\partial}{\partial x_{3n+i}}\right) = \frac{\partial}{\partial x_{5n+i}} & J_2\left(\frac{\partial}{\partial x_{3n+i}}\right) = \frac{\partial}{\partial x_{6n+i}} & J_3\left(\frac{\partial}{\partial x_{3n+i}}\right) = -\frac{\partial}{\partial x_i} \\
J_1\left(\frac{\partial}{\partial x_{4n+i}}\right) = -\frac{\partial}{\partial x_{2n+i}} & J_2\left(\frac{\partial}{\partial x_{4n+i}}\right) = \frac{\partial}{\partial x_{n+i}} & J_3\left(\frac{\partial}{\partial x_{4n+i}}\right) = \frac{\partial}{\partial x_{7n+i}} \quad \rightarrow \\
J_1\left(\frac{\partial}{\partial x_{5n+i}}\right) = -\frac{\partial}{\partial x_{3n+i}} & J_2\left(\frac{\partial}{\partial x_{5n+i}}\right) = -\frac{\partial}{\partial x_{7n+i}} & J_3\left(\frac{\partial}{\partial x_{5n+i}}\right) = \frac{\partial}{\partial x_{n+i}} \\
J_1\left(\frac{\partial}{\partial x_{6n+i}}\right) = \frac{\partial}{\partial x_{7n+i}} & J_2\left(\frac{\partial}{\partial x_{6n+i}}\right) = -\frac{\partial}{\partial x_{3n+i}} & J_3\left(\frac{\partial}{\partial x_{6n+i}}\right) = \frac{\partial}{\partial x_{2n+i}} \\
J_1\left(\frac{\partial}{\partial x_{7n+i}}\right) = -\frac{\partial}{\partial x_{6n+i}} & J_2\left(\frac{\partial}{\partial x_{7n+i}}\right) = \frac{\partial}{\partial x_{5n+i}} & J_3\left(\frac{\partial}{\partial x_{7n+i}}\right) = -\frac{\partial}{\partial x_{4n+i}}
\end{array} \tag{11}$$

A canonical local basis $\{J_1^* \,, J_2^* \,, J_3^*\}\ of\ V^*$ of the cotangent space $T^*(M)$ of manifold $M$ satisfies the following condition as follows:

$$J_1^{*2} = J_2^{*2} = J_3^{*2} = J_1^* J_2^* J_3^{*2} J_2^* J_1^* = -I \,, \quad \rightarrow \tag{12}$$

Defining by

$$\begin{array}{lll}
J_1^*(dx_i) = dx_{n+i} & J_2^*(dx_i) = dx_{2n+i} & J_3^*(dx_i) = dx_{3n+i} \\
J_1^*(dx_{n+i}) = -dx_i & J_2^*(dx_{n+i}) = -dx_{4n+i} & J_3^*(dx_{n+i}) = -dx_{5n+i} \\
J_1^*(dx_{2n+i}) = dx_{4n+i} & J_2^*(dx_{2n+i}) = -dx_i & J_3^*(dx_{2n+i}) = -dx_{6n+i} \\
J_1^*(dx_{3n+i}) = dx_{5n+i} & J_2^*(dx_{3n+i}) = dx_{6n+i} & J_3^*(dx_{3n+i}) = -dx_i \\
J_1^*(dx_{4n+i}) = -dx_{2n+i} & J_2^*(dx_{4n+i}) = dx_{n+i} & J_3^*(dx_{4n+i}) = dx_{7n+i} \quad \rightarrow \\
J_1^*(dx_{5n+i}) = -dx_{3n+i} & J_2^*(dx_{5n+i}) = -dx_{7n+i} & J_3^*(dx_{5n+i}) = dx_{n+i} \\
J_1^*(dx_{6n+i}) = dx_{7n+i} & J_2^*(dx_{6n+i}) = -dx_{3n+i} & J_3^*(dx_{6n+i}) = dx_{2n+i} \\
J_1^*(dx_{7n+i}) = -dx_{6n+i} & J_2^*(dx_{7n+i}) = dx_{5n+i} & J_3^*(dx_{7n+i}) = -dx_{4n+i}
\end{array} \tag{13}$$

**Definition Standard Cliffordian $K\ddot{a}hler$ Manifolds:** A canonical local basis $\{J_1, J_2, J_3\}\ of\ V$ of the tangent space $TM$ of Manifold M and A canonical local basis $\{J_1^* \,, J_2^* \,, J_3^*\}\ of\ V^*$ of the cotangent space $T^*M$ is called Standard Cliffordian $K\ddot{a}hler$ Manifolds and satisfies Lagrangian and Hamiltonian Mechanical systems.

# CONCLUSION

**In this paper we conclude that there are two types of standard cliffordian $K\ddot{a}hler$ manifolds:**

**The first type:** The Standard Cliffordian $K\ddot{a}hler$ manifolds on a canonical Structure of the tangent bundle .
**The second type:** The Standard Cliffordian $K\ddot{a}hler$ manifolds on a canonical Structure of the cotangent bundle $T^*M$.
Thus, we find that the canonical structure on the tangent bundle of the Standard Cliffordian $K\ddot{a}hler$ manifolds satisfies Lagrangian dynamical systems. And the a canonical structure on the cotangent bundle of the Standard Cliffordian $K\ddot{a}hler$ manifolds satisfies Hamiltonian dynamical systems.

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