Available online at http://www.journalijdr.com

IJDR

ISSN: 2230-9926

International Journal of Development Research
Vol. 16 Issue, 03, pp. 70068-70071, March, 2026
https://doi.org/10.37118/ijdr.30662.03.2026

RESEARCH ARTICLE OPEN ACCESS

CUBE ADDITION GRAPH: STRUCTURE AND CONNECTIVITY

Nidhi Khandelwal*, Pravin Garg and Ravi Ratn Gaur

Department of Mathematics, University of Rajasthan, Jaipur-302004, Rajasthan, India

ARTICLE INFO

Article History:
Received 14th December, 2025
Received in revised form
26th January, 2026
Accepted 11th February, 2026
Published online 30th March, 2026

Key Words:
Cayley Sum Graph, Commutative Ring, Finite Field, Connectivity, Regularity, Polynomial Ring, Quotient Ring.

****Correspondingauthor:***
Nidhi Khandelwal

ABSTRACT

We introduce the addition cube graph over a ring R, whose vertices are the elements of R, and two distinct vertices $x$and $y$are adjacent whenever $x+y$is a cube in R. We investigate fundamentalgraph-theoreticproperties including degree, regularity, connectivity, bipartiteness, and Hamiltonian paths.For finite fields, we determine conditions under which every element is a cube and analyze the resulting connectivity behavior. In particular, the graph is complete over $\mathbb{R}$and $\mathbb{C}$, connected over $\mathbb{Z}$and$\mathbb{Z}_n$, and disconnected over $F[x]$when char $(F) = 3$.We further examine relationships between the graphs of a ring, its ideals, and quotient rings, proving that connectivity of both $AC(R/I)$and $AC(I)$implies the connectivity of $AC(R)$.

Citation: Nidhi Khandelwal, Pravin Garg and Ravi Ratn Gaur. 2026. "Cube Addition Graph: Structure and Connectivity". *International Journal of Development Research*, 16, (03), 70068-70071.

INTRODUCTION

The study of graphs associated with algebraic structures has become an active research area connecting algebra and graph theory. The basic idea is to associate a graph to a ring in such a way that algebraic properties of ring elements are reflected in combinatorial properties of the graph. A pioneering contribution in this direction was made by Beck [4], who introduced the zero–divisor graph of a commutative ring.Later, Anderson and Livingston [2] systematically investigated
Its structure and proved that the graph encodes many important ring–theoretic information. Since then, numerous algebraic graphs on rings have been introduced and studied. These include ideal-based zero–divisor graphs [9], annihilating–ideal graphs [5], total graphs [1], comaximal graphs, and unit graphs [3]. In the past few years, algebraically defined graphs on rings and finite fields have continued to be investigated. Recent works (see, [8]) studied connectivity, domination, and Hamiltonian properties of graphs associated with algebraic subsets of rings and finite fields. These results demonstrate that the arithmetic behavior of specific subsets such as units, squares, and powers plays a decisive role in determining graph-theoretic properties. Square element graphs of commutative rings were introduced [6] and further investigated in [7]. The addition cube graph considered in this paper may be viewed as a higher power analogue obtained by replacing square elements with cube elements in the adjacency condition. Our goal is to investigate how the algebraic properties of cubic elements influence the structure of the associated graph.

Definition 1. Let Rbe a ring and define$C = \{a^3 : a \in R\}$ to be the set of cube elements of R. The addition cube graph of R, denoted by $AC(R)$, is the simplegraph whose vertex set is$V(AC(R)) = R$,and two distinct vertices $x, y \in R$are adjacent if and only if $x + y \in C$.The set $C$is closed under additive inverses. Moreover, if $R$is a commutative ring, then $C$is closed under multiplication.If Rhas unity 1, then $1 \in C$. For distinct vertices $, y \in V$, we write$x \leftrightarrow y$ if and only if $x + y \in C$.

Ring Theoretic Properties

Proposition 1. Let Rbe a commutative ring and let$C = \{a^3 : a \in R\}$be the set of cube elements of R. If $R$is a ring of characteristic 3, then $C$is a subring of R.

Proof. First, note that $0 = 0^3 \in C$, hence Ccontains the additive identity. For any $a \in R$,

$-(a^3) = (-a)^3$,

so $C$is closed under additive inverses. Since $R$is commutative,

$a^3 b^3 = (ab)^3$,

and therefore $C$is closed under multiplication. It remains to show closure under addition. For $a, b \in R$,

$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$.
Hence

$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$.

If char $(R) = 3$, then the second term vanishes and we obtain

$$a^3 + b^3 = (a+b)^3 \in C.$$

Thus $C$is closed under addition. Therefore $C$is a subring of R.

Theorem 2. Let Rbe a commutative ring with unity and let $C = a^3: a \in R\}$. If$C$is a subring of R. Then Chas the same characteristic as R; that is,
char $(C) =$ char (R).

Proof. Since $1_R \in C$, the subring Ccontains the unity of R. Hence

$1_C = 1_R$.

Let $n =$ char (R). Then

$n \cdot 1_R = 0$in R.

Because $1_C = 1_R$, we also have

$n \cdot 1_C = 0$in C.

Therefore, char (C)divides n.

Conversely, suppose char $(C) = m$. Then

$m \cdot 1_C = 0$in C.

Since $1_C = 1_R$, this implies

$m \cdot 1_R = 0$in R.

By the minimality of $n =$ char (R), we must have $n \mid m$. Hence
$m \mid n$and $n \mid m$, and therefore

char $(C) =$ char (R).

Theorem 3. If $F$is a finite field with order $m$and 3divides m, then every element of $F$is a cube element.

Proof. Let $0 \neq a \in F$. Then a^{-1}exists and

$a^{m-1} = 1 \;\Rightarrow\; a = a \cdot a^{m-1} = a^m = (a^{m/3})^3 = y^3$,

where $y = a^{m/3}$.Thus, every element of $F$is a cube element.

Graphical Structure of Addition Cube Graph

Definition 2. Let Gbe an abelian group and let $S \subseteq G$. The addition Cayley graph of $G$with respect to S, denoted by $Cay^+(G,S)$, is the simple graph with vertex set Gin which two distinct vertices $x, y \in G$are adjacent if and only if $x + y \in S$. In the literature, addition Cayley graphs are also called addition graphs, Cayley sum graphs and sum graphs.

Theorem 4. Let Rbe a commutative ring and let$C = \{a^3: a \in R\}$.Then the addition cube graph $AC(R)$is the addition Cayley graph of the additive group $(R,+)$with respect to C, i.e.,

$$AC(R) = Cay^+((R,+),C).$$

Proof. The vertex set of $AC(R)$is R. By definition, two distinct vertices $x, y \in R$are adjacent in $AC(R)$if and only if

$$x + y \in C.$$

On the other hand, the addition Cayley graph $Cay^+\big((R,+),C\big)$has vertex set $(R,+)$and two distinct vertices $x, y \in R$are adjacent if and only if

$$x + y \in C.$$

Since both the adjacency conditionsare identical, both graphs have the same vertex set and the same edge set. Hence

$AC(R) = Cay^+((R,+),C)$.

Proposition 5. The graph $AC(R)$is undirected.

Proof. Let $x, y \in R$with $x + y \in C$. Then there exists $a \in R$such that

$x + y = a^3$.

Since $(R,+)$is an abelian group, hence

$y + x = a^3 \in C$.

Therefore $y + x \in C$, which implies that $x \leftrightarrow y$if and only if $y \leftrightarrow x$. Thus $AC(R)$is an undirected graph.

Theorem 6. Let Fbe a finite field with char $(F) > 2$and $C = \{a^3: a \in F\}$. Then the addition cube graph $AC(F)$is not bipartite.

Proof. Since char $(F) > 2$, so $1 \neq -1$and $0 = 0^3$, $1 = 1^3$,$-1 = (-1)^3$, we have $0, 1, -1 \in C$. Now,

$0 + 1 = 1 \in C, 1 + (-1) = 0 \in C, (-1) + 0 = -1 \in C$.

Hence$0 \leftrightarrow 1, 1 \leftrightarrow -1, -1 \leftrightarrow 0$.

So, the vertices $\{0,1,-1\}$form a triangle. Therefore $AC(F)$contains an odd cycle and is not bipartite.

Theorem 7. Let Rbe a finite commutative ring and let $AC(R)$be its addition cube graph. Then for any vertex $x \in R$,

$$\deg(x) = \begin{cases} |C| - 1, & \text{if } 2x \in C, \\ |C|, & \text{if } 2x \notin C, \end{cases}$$

where $C = \{a^3: a \in R\}$.

Proof.A vertex $y$is adjacent to $x$if and only if $x + y \in C$. Thus, there exists $c \in C$such that

$x + y = c \;\Leftrightarrow\; y = c - x$.

Hence every neighbor of $x$is of the form $y = c - x$for some $c \in C$.Define a map $\phi: C \to R$by $\phi(c) = c - x$.If $\phi(c_1) = \phi(c_2)$, then

$c_1 - x = c_2 - x \;\Longrightarrow\; c_1 = c_2$.

Thus, $\phi$is injective, and distinct elements of Cproduce distinct vertices y. Therefore, $x$has exactly $|C|$possible neighbors.

The equality $y = x$occurs precisely when

$c - x = x \;\Leftrightarrow\; c = 2x$.

Since loops are not allowed in $AC(R)$, this vertex must be excluded whenever $2x \in C$.

Hence one neighbor is removed exactly when $2x \in C$, giving

$$\deg(x) = \begin{cases} |C| - 1, & if\ 2x \in C \\ |C|, & if\ 2x \notin C. \end{cases}$$

Theorem 8. Let Rbe a finite commutative ring and let $AC(R)$be its addition cube graph. Then $AC(R)$is regular if and only if either $2R \subseteq C$or $2R \cap C = \emptyset$, where $C = \{a^3: a \in R\}$.

Proof. From the degree formula, for every vertex $x \in R$,

$$\deg(x) = \begin{cases} |C| - 1, & if\ 2x \in C \\ |C|, & if\ 2x \notin C. \end{cases}$$

Thus, the degree of a vertex depends only on whether $2x$belongs to C. Suppose $AC(R)$is regular. Then all vertices have the same degree.If there exist $x, y \in R$such that $2x \in C$and $2y \notin C$, then

$$\deg(x) = |c| - 1 \text{and} \deg(y) = |C|,$$

which contradicts the regularity of $AC(R)$. Hence this cannot occur. Therefore, either $2x \in C$for all $x \in R$, or $2x \notin C$for all $x \in R$. Equivalently, either $2R \subseteq C$or $2R \cap C = \emptyset$.

Conversely, if $2R \subseteq C$, then $2x \in C$for every $x \in R$, and hence

$$\deg(x) = |C| - 1$$

for all vertices x. Thus $AC(R)$is regular.

If $2R \cap C = \emptyset$, then $2x \notin C$for every $x \in R$, and hence

$$\deg(x) = |C|$$

for all vertices x. Again $AC(R)$is regular. Therefore $AC(R)$is regular if and only if $2R \subseteq C$or $2R \cap C = \emptyset$.

Theorem 9. For every integer $n \geq 2$, the addition cube graph $AC(\mathbb{Z}_n)$is connected.

Proof. Let $C = \{t^3 : t \in \mathbb{Z}_n\}$. Since $0 = 0^3$and $1 = 1^3$, we have $0,1 \in C$. Hence two vertices $x, y \in \mathbb{Z}_n$are adjacent whenever

$x + y \equiv 0(\text{mod}n)orx + y \equiv 1(\text{mod}n)$.

We first show that all nonzero vertices lie in one component. For $1 \leq k \leq \lfloor n/2 \rfloor$,

$k + (n - k) = n \equiv 0(\text{mod}n)$,

so $k \leftrightarrow (n - k)$. For $1 \leq k \leq \lfloor n/2 \rfloor$,

$(n - k) + (k + 1) = n + 1 \equiv 1(\text{mod}n)$,

so $(n - k) \leftrightarrow (k + 1)$. Thus, we obtain a path

$1 \leftrightarrow (n - 1) \leftrightarrow 2 \leftrightarrow (n - 2) \leftrightarrow 3 \leftrightarrow \cdots$,

which visits every vertex in $\mathbb{Z}_n \setminus \{0\}$. Therefore, all nonzero vertices belong to a single connected component. Finally, $0 + 1 = 1 \in C \Rightarrow 0 \leftrightarrow 1$. Since 1lies in the component containing all nonzero vertices, the vertex 0also belongs to this component. Hence every vertex of $\mathbb{Z}_n$is reachable from every other vertex, and therefore $AC(\mathbb{Z}_n)$is connected.

Remark 10. The addition cube graph $AC(\mathbb{Z}_n)$contains a Hamiltonian path.

Theorem 11. The addition cube graph $AC(\mathbb{Z})$is connected, where $\mathbb{Z}$is the ring of integers.

Proof. Let $C = \{a^3 : a \in \mathbb{Z}\}$. Since$0 = 0^3and1 = 1^3$,we have $0,1 \in C$. For any integer $r \in \mathbb{Z}$,

$r + (-r) = 0 \in C \Rightarrow r \leftrightarrow -r$,
and

$$(-r) + (r + 1) = 1 \in C \Rightarrow -r \leftrightarrow r + 1.$$

Hence there is a path

$$r \leftrightarrow -r \leftrightarrow r + 1.$$

Thus, repeatedly applying this step,

$0 \leftrightarrow 1 \leftrightarrow -1 \leftrightarrow 2 \leftrightarrow -2 \leftrightarrow 3 \leftrightarrow -3 \leftrightarrow \cdots \leftrightarrow r \leftrightarrow -r \leftrightarrow r + 1 \leftrightarrow \cdots$
Therefore, all vertices lie in a single connected component, and hence $AC(\mathbb{Z})$is connected.

Theorem 12. If $\mathbb{R}$is the ring of real numbers, then the addition cube graph $AC(\mathbb{R})$is a complete graph.

Proof. Let $C = \{t^3 : t \in \mathbb{R}\}$. Every real number has a real cube root, hence for each $r \in \mathbb{R}$there exists $a \in \mathbb{R}$such that $r = a^3$. Thus $C = \mathbb{R}$. For any distinct vertices $x, y \in \mathbb{R}$, we have $x + y \in \mathbb{R} = C$, and therefore $x \leftrightarrow y$. Hence every pair of vertices is adjacent, and $AC(\mathbb{R})$is complete.

Remark 13. The same argument shows that $AC(\mathbb{C})$is also a complete graph, where $\mathbb{C}$is the ring of complex numbers.

Theorem 14. If $F$is a finite field with order $m$and 3divides m, then $AC(F)$is a complete graph.

Proof. From Theorem 3, all the elements of field $F$are cube element, so every two distinct elements of $F$are adjacent in $AC(F)$. Hence $AC(F)$is a complete graph.

Theorem 15. Let $k > 1$and $R = k\mathbb{Z}$. Then the addition cube graph $AC(k\mathbb{Z})$is disconnected.

Proof. In $R = k\mathbb{Z}$, every element has the form $a = km$with $m \in \mathbb{Z}$, and hence

$a^3 = (km)^3 = k^3 m^3$.

Let $x = ku$and $y = kv$be vertices of $AC(k\mathbb{Z})$with $u, v \in \mathbb{Z}$. By definition of adjacency,

$x \leftrightarrow y \iff x + y = a^3$ for some $a \in R$
$\iff ku + kv = k^3 m^3$
$\iff u + v = k^2 m^3$, for some $m \in \mathbb{Z}$.

Consequently,

$$v \equiv -u(\text{mod}k^2). \qquad (1)$$

Assume there exists a path starting from the vertex $k = k \cdot 1$and let vertices of path be $c_i = ku_i$. From (1), if $c_{i-1} \leftrightarrow c_i$, then

$u_i \equiv -u_{i-1}(\text{mod}k^2)$.

Starting with $u_0 = 1$, an easy induction gives

$u_i \equiv (-1)^i(\text{mod}k^2)$.

Hence every vertex reachable from $k$satisfies $u \equiv \pm 1(\text{mod}k^2)$. However

$-k^3 = k(-k^2)$

corresponds to $u = -k^2 \equiv 0(\text{mod}k^2)$, which does not satisfy the above condition. Thus, no path exists between $k$and $-k^3$, and therefore $AC(k\mathbb{Z})$is disconnected.

Theorem 16. Let Rbe a commutative ring with unity and Ian ideal of R. If the addition cube graphs $AC(I)$and $AC(R/I)$are connected, then $AC(R)$is connected.
Proof. Recall that in $AC(R)$two vertices $x, y \in R$are adjacent if and only if $x + y = a^3$, for some $a \in R$.

Step 1: Every element of $R$is connected to an element of I.

Let $a \in R$. Since $AC(R/I)$is connected, there exists a path

$$a + I = c_0 + I \leftrightarrow c_1 + I \leftrightarrow \cdots \leftrightarrow c_r + I = I$$

in $AC(R/I)$. We shall construct a path in $AC(R)$starting from $a$and ending at an element of I.

Set $d_0 = a$. We construct elements $d_1, d_2, \ldots, d_r \in R$inductively such that $d_j \in c_j + I$. Consider $d_{j-1} \in c_{j-1} + I$. Then

$$d_{j-1} = c_{j-1} + s \text{ for some } s \in I. \qquad (2)$$

Since $c_{j-1} + I \leftrightarrow c_j + I$in $AC(R/I)$, we have

$$c_{j-1} + c_j = b^3 + t, \text{ for some } b \in R, t \in I \qquad (3)$$

Define$i_j = t + s \in I$, $d_j = c_j - i_j$. Clearly$d_j \in c_j + I$. Now compute:

$$\begin{aligned} d_{j-1} + d_j &= (c_{j-1} + s) + (c_j - i_j) \\ &= (c_{j-1} + c_j) + (s - i_j) \\ &= (b^3 + t) + (s - (t + s)) \\ &= b^3. \end{aligned}$$

Thus $d_{j-1} \leftrightarrow d_j$in $AC(R)$.

Repeating this construction for $j = 1,2, \ldots, r$, we obtain a path

$$a = d_0 \leftrightarrow d_1 \leftrightarrow \cdots \leftrightarrow d_r$$

in $AC(R)$. Since $c_r + I = I$, we have $c_r \in I$. Because $i_r \in I$,

$$d_r = c_r - i_r \in I.$$

Hence every element $a \in R$is connected to some element of I.

Step 2: Connectivity of $AC(R)$.

Let $x, y \in R$. From Step 1, $x$is connected to some $i_1 \in I$and $y$is connected to some $i_2 \in I$. Because $AC(I)$is connected, there exists a path within Ijoining $i_1$and i_2. Concatenating these paths yields a path from $x$to $y$in $AC(R)$.

Therefore $AC(R)$is connected.

Theorem 18. Let Fbe a field with char $(F) = 3$. Then the addition cube graph $AC(F[x])$is disconnected.

Proof. In $AC(F[x])$, two polynomials $f, g \in F[x]$are adjacent if and only if$f + g = h^3$for some $h \in F[x]$.

Since char $(F) = 3$, the Frobenius identity holds:

$$(a + b)^3 = a^3 + b^3 \text{ for all } a, b \in F[x].$$

Let$h(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m \in F[x]$. Repeated use of the identity $(a + b)^3 = a^3 + b^3$gives

$$h(x)^3 = b_0^3 + b_1^3 x^3 + b_2^3 x^6 + \cdots + b_m^3 x^{3m}.$$

Hence every monomial appearing in a cube polynomial has degree divisible by 3.

Step 1: Cube and non-cube polynomials cannot be adjacent. Let $f(x)$be a cube polynomial and let $g(x)$be a polynomial that contains a monomial$ax^m$with$3 \nmid m$.Every term of $f(x)$has degree divisible by 3, so no cancellation can remove the term ax^min $f(x) + g(x)$.

Therefore $f(x) + g(x)$still contains a term whose degree is not divisible by 3, and hence $f(x) + g(x)$cannot be a cube polynomial. Thus, a cube polynomial cannot be adjacent to a non-cube polynomial in $AC(F[x])$.

Step 2: No path between cube and non-cube vertices.

Suppose there exists a path

$$f_0 \leftrightarrow f_1 \leftrightarrow \cdots \leftrightarrow f_r$$

in $AC(F[x])$with f_0a cube polynomial. By Step 1, if $f_{j-1} \leftrightarrow f_j$, then $f_j$must also be a cube polynomial. By induction, every vertex $f_j$in the path is a cube polynomial. Hence a non-cube polynomial cannot be reached from a cube polynomial. So, the set of vertices $F[x]$splits into two nonempty disjoint subsets $\{\text{cube polynomials}\}$and $\{\text{non-cube polynomials}\}$, and there are no edges between them. Therefore $AC(F[x])$is disconnected.

CONCLUSION

We introduced the addition cube graph of a commutative ring R, in which two vertices are adjacent whenever their sum is a cube element of R. We studied basic properties of this graph, including degree, regularity, connectivity, bipartiteness, and Hamiltonian paths, and showed how these properties depend on the algebraic structure of the ring. In particular, we obtained criteria for regularity, proved that connectivity can be lifted from an ideal $I$and the quotient ring $R/I$to R, and analyzed several important examples. The graph is connected for $\mathbb{Z}$and $\mathbb{Z}_n$, complete for $\mathbb{R}$and $\mathbb{C}$, and disconnected for $F[x]$when char $(F) = 3$. These results highlight the interplay between ring-theoretic properties and the structure of the associated addition graph and suggest further study of related power-based addition graphs.

REFERENCES

Anderson D.F. & Badawi A. 2008.The total graph of a commutative ring.*Journal of Algebra*, 320, 2706-2719.

AndersonD. F. & LivingstonP. S. 1999 The zero-divisor graph of a commutative ring. *Journal of Algebra*, 217 (2), 434–447.

AshrafiN., Maimani H. R., Pournaki M. R., & Yassemi, S. (2010). Unit Graphs Associated with Rings. *Communications in Algebra*. *38*(8), 2851–2871.

BeckI. 1988, Coloring of commutative rings.*Journal of Algebra* 116 (1), 208–226.

Behboodi M. &Rakeei Z. 2012. The annihilating-ideal graph of commutative rings I. *Journal of Algebra and its applications*, 10(4), 727-739.

GuptaR. S. &SenM.K. 2015, The Square element graph over a finite commutative ring.Southeast Asian Bulletin of Mathematic, 39, 407–428.

GuptaR. S. &SenM.K. 2017, The Square element graph over a ring.Southeast Asian Bulletin of Mathematic, 41, 663-682.

Madhumitha, S., &Naduvath, S. 2023. Graphs Defined on Rings: A Review. *Mathematics*, *11*(17), 3643.

Redmond, S. P. (2003). An Ideal-Based Zero-Divisor Graph of a Commutative Ring. *Communications in Algebra*, *31*(9), 4425–4443.

****